

Computation of framed deformation functors

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Abstract

In this work we compute the framed deformation functor associated to a reducible representation given as direct sum of 2-dimensional representations associated to elliptic curves with appropriate local conditions. Such conditions arise in the works of Schoof and correspond to reduction properties of modular elliptic curves.

1 Introduction

This work originates from the articles of Schoof about classification of abelian varieties [15]. There he examines the case of abelian varieties over \mathbb{Q} with semistable reduction in only one prime ℓ and good reduction everywhere else and proves that they do not exist for $\ell = 2, 3, 5, 7, 13$, while for $\ell = 11$ they are given by products of the Jacobian variety $J_0(11)$ of the modular curve $X_0(11)$. In [16] he makes some generalisations of this result when ℓ is not a prime and the base field is not \mathbb{Q} , but a quadratic field. Some similar results, given in terms of p -divisible groups, were also previously obtained by Abrashkin in [1].

The main purpose of this work is to translate some results of those articles in terms of deformation theory of representations associated to elliptic curves. We examine the following setting: let $p \neq \ell$ be distinct primes and $S = \{p, \ell, \infty\}$. Let $\bar{\rho}_i : G_S \rightarrow GL_2(k)$ $i = 1, \dots, n$ be Galois representations, where k is a finite field of characteristic p and G_S is the Galois group of the maximal extension of \mathbb{Q} unramified outside S . We can suppose that there are exactly r non-isomorphic representations among them and, up to reordering indexes, suppose that they are $\bar{\rho}_1, \dots, \bar{\rho}_r$. Then we can write

$$\bar{\rho} = \bar{\rho}_1 \oplus \dots \oplus \bar{\rho}_n = \oplus_{i=1}^r \bar{\rho}_i^{e_i} : G_S \rightarrow GL_{2n}(k) \quad (1)$$

where $\sum_{i=1}^r e_i = n$. The main result is the following

Theorem 1.1. : *Suppose that:*

1. *the k -vector space $Ext_{\underline{D}, p}^1(V_{\bar{\rho}_i}, V_{\bar{\rho}_j})$ of killed-by- p extensions is trivial for every $i, j = 1, \dots, r$;*

$$2. \operatorname{Hom}_G(V_{\bar{\rho}_i}, V_{\bar{\rho}_j}) = \begin{cases} k & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

Let $F_{\underline{D}}^{\square}$ be the framed deformation functor associated to $\bar{\rho}$ with the local conditions:

- ρ is p -flat over $\mathbb{Z}[1/\ell]$;
- ρ satisfies $(\rho_i(g) - \operatorname{Id})^2 = 0$ for every $g \in I_{\ell}$;
- ρ is odd.

Then $F_{\underline{D}}^{\square}$ is represented by a framed universal ring $R_{\underline{D}}^{\square}$ which is isomorphic to $W(k)[[x_1, \dots, x_N]]$, where $N = 4n^2 - \sum_{i=1}^r e_i^2$.

The setting works in particular when $\bar{\rho}_i$ is the representation associated to the p -torsion points of an elliptic curve E_i over \mathbb{Q} having semistable reduction in ℓ and good supersingular reduction at p , as the varieties described in [15]. Moreover the local condition in ℓ corresponds to the condition of semistable action described in [15, Section 2], while the condition in p is the classical flatness condition introduced in [13]. The final result want to express that the framed deformation ring turns out to be the “simplest” possible, giving an analog for deformation of [15, Th. 8.3].

The work is structured as such: in section 2 we recall the main features of deformation theory, following mainly Mazur’s original formulation (see [11, 12]) and we introduce framed deformation functors, following Kisin’s approach [9], but avoiding the use of the formalism of groupoids. Then in the following three chapters we describe the local conditions we have used in our theorem: first we deal with the flatness condition in the residual prime p , which is treated according the initial results of Ramakrishna [13] and expanded by the work of Conrad [3, 4, 5]; then we pass to examine the semistability condition, which is treated as a particular case of representation of Steinberg type: the computation of the universal ring in this part is pretty indirect and passes through the use of formal schemes [10]. Finally, for the archimedean prime, the computation of the deformation ring is performed directly. Chapters 6 and 7 are dedicated to local-to-global arguments, which consent to build up a presentation of a global deformation ring using the computations on the local ones we did in the previous chapters. The definition of geometric deformation functor is introduced, too. It is due to Kisin [10] and the name comes from the “geometric” representations defined in Serre’s conjecture. In the final chapter Theorem 1.1 is proved with the use of the technical instruments introduced and through a final direct computation of the universal framed deformation, using matrix algebras.

2 Basics Notions of Deformation theory

Let k be a finite field of characteristic p and let S be a finite set of rational primes containing p and the archimedean prime. Denoting by \mathbb{Q}_S the maximal extension of \mathbb{Q} unramified outside S and by $G_S = \text{Gal}(\mathbb{Q}_S/\mathbb{Q})$, consider a Galois representation $\bar{\rho} : G_S \rightarrow GL_N(k)$ and denote by $V_{\bar{\rho}}$ the associated G_S -module. Let A be a complete noetherian local $W(k)$ -algebra with residue field k and denote by $\hat{\mathcal{A}}r$ the category of such algebras. A *lift* of $\bar{\rho}$ to A is a representation $\rho_A : G_S \rightarrow GL_N(A)$ such that the diagram

$$\begin{array}{ccc} & & GL_N(A) \\ & \nearrow \rho_A & \downarrow \pi_A \\ G_S & \xrightarrow{\bar{\rho}} & GL_N(k) \end{array}$$

commutes, where π_A is the natural projection on the residue field. Two lifts ρ_1, ρ_2 are said to be *equivalent* if there exists a matrix $M \in \text{Ker}(\pi_A)$ such that $M\rho_1(g)M^{-1} = \rho_2(g)$ for every $g \in G_S$.

A *deformation* of $\bar{\rho}$ to A is an equivalence class of lifts. The starting representation $\bar{\rho}$ is called *residual*.

Deformations can be better understood via a categorical approach. Given a Galois representation $\bar{\rho}$, we can define the *deformation functor*

$$F_{\bar{\rho}} : \hat{\mathcal{A}}r \rightarrow \underline{\text{Sets}}, \quad (2)$$

which associates to an element $A \in \hat{\mathcal{A}}r$ the set of deformation classes of $\bar{\rho}$ to A .

Kisin has introduced a variant of the deformation functor. Let β be a k -basis of the Galois module $V_{\bar{\rho}}$. A *framed deformation* of the couple $(V_{\bar{\rho}}, \beta)$ to a ring $A \in \hat{\mathcal{A}}r$ is a couple (V_A, β_A) , where V_A is a free N -dimensional A -module with continuous G_S -action lifting the action of $V_{\bar{\rho}}$ and β_A is an A -basis of V_A lifting β . We can then define the *framed deformation functor* $F_{\bar{\rho}}^{\square} : \hat{\mathcal{A}}r \rightarrow \underline{\text{Sets}}$ which associates to an algebra $A \in \hat{\mathcal{A}}r$ the set of framed deformation classes of $(V_{\bar{\rho}}, \beta)$ to A .

Theorem 2.1. 1. *The framed deformation functor is representable by a ring $R^{\square} = R_{\bar{\rho}}^{\square} \in \hat{\mathcal{A}}r$.*

2. *If $\bar{\rho}$ satisfies the trivial centralizer condition $\text{End}_{k[G]}(V_{\bar{\rho}}) \simeq k$, then $F_{\bar{\rho}}$ is representable by a ring $R = R_{\bar{\rho}} \in \hat{\mathcal{A}}r$.*

The rings R and R^{\square} are called the *universal deformation ring* and the *universal framed deformation ring* of $\bar{\rho}$ respectively. They are universal in the sense that any deformation of $\bar{\rho}$ to an element $A \in \hat{\mathcal{A}}r$ can be recovered via a unique homomorphism $R \rightarrow A$.

Among the algebras in $\hat{\mathcal{A}}r$ there is one with particular properties. Let $k[\epsilon]$ be the ring of polynomials in ϵ with the condition $\epsilon^2 = 0$ and let F be a deformation functor. The *tangent space* of F is the set $F(k[\epsilon])$. It has a natural structure of k -vector space.

Proposition 2.2. 1. $F_{\bar{\rho}}(k[\epsilon])$ is a finite dimensional k -vector space;

2. $F_{\bar{\rho}}(k[\epsilon]) \simeq H^1(G_S, \text{Ad}(\bar{\rho})) \simeq \text{Ext}_{k[G]}^1(V_{\bar{\rho}}, V_{\bar{\rho}})$ as k -vector spaces;

3. $\dim_k F_{\bar{\rho}}^{\square}(k[\epsilon]) = \dim_k F_{\bar{\rho}}(k[\epsilon]) + N^2 - \dim_k H^0(G_S, \text{Ad}(\bar{\rho}))$.

Let $F_{\bar{\rho}}$ be a deformation functor and let \underline{P} be the category of pairs (A, V_A) with $A \in \hat{\underline{A}}r$ and $V_A \in F_{\bar{\rho}}(A)$. Let \underline{D} be a full subcategory of \underline{P} . We say that \underline{D} is a *deformation condition* if the following conditions hold:

1. if $(A, V_A) \rightarrow (B, V_B)$ is a morphism in \underline{P} and $(A, V_A) \in \underline{D}$, then $(B, V_B) \in \underline{D}$;
2. if $(A, V_A) \rightarrow (B, V_B)$ is an injective morphism in \underline{P} and $(B, V_B) \in \underline{D}$, then $(A, V_A) \in \underline{D}$;
3. if $(A \times_C B, V)$ lies in \underline{D} , then also the projections (A, V_A) and (B, V_B) do.

Given a deformation condition \underline{D} , we can consider the functor $F_{\bar{\rho}, \underline{D}} : \hat{\underline{A}}r \rightarrow \underline{Sets}$ that sends a ring $A \in \hat{\underline{A}}r$ to the set of deformations ρ of $\bar{\rho}$ to A such that $(A, V_{\rho}) \in \underline{D}$. If $F_{\bar{\rho}}$ is representable by a ring $R_{\bar{\rho}}$, then $F_{\bar{\rho}, \underline{D}}$ is representable, too, by a quotient of $R_{\bar{\rho}}$. Moreover the tangent space $F_{\bar{\rho}, \underline{D}}(k[\epsilon])$ is a k -vector subspace of $F_{\bar{\rho}}(k[\epsilon])$.

Given a Galois representation $\bar{\rho} : G_S \rightarrow GL_N(k)$ a natural way to attach a deformation condition is the following: for every $\ell \in S$ we consider the restriction $\bar{\rho}_{\ell} : G_{\ell} \rightarrow GL_N(k)$ together with a local deformation condition \underline{D}_{ℓ} . Then we can define a global deformation condition \underline{D} given by the objects $(A, V_A) \in \underline{P}$ whose local restriction to ℓ lies in \underline{D}_{ℓ} for every $\ell \in S$.

Another important example of deformation condition is given by the fixed determinant. Let $\chi : G_S \rightarrow W(k)^*$ be a linear character. We say that a representation $\rho : G_S \rightarrow GL_N(A)$ has *determinant* χ if the G_S -action induced on the wedge product $\Lambda^N(V_{\rho})$ is given by the character

$$\chi_A : G_S \xrightarrow{\chi} W(k)^* \xrightarrow{\phi} A^*, \quad (3)$$

where ϕ is the restriction of the $W(k)$ -algebra structure morphism. The subcategory \underline{D} of pairs (A, V_{ρ}) such that ρ has determinant χ is a deformation condition and the corresponding deformation functor is denoted as $F_{\bar{\rho}}^{\chi}$. Moreover we have that

$$F_{\bar{\rho}}^{\chi}(k[\epsilon]) \simeq H^1(G_S, \text{Ad}^0(\bar{\rho})), \quad (4)$$

where Ad^0 denotes the vector space of matrices with trace zero provided by the adjoint G_S -action.

3 The local flat deformation functor

In this section we want to deal with a local deformation condition which refers to the prime p , characteristic of the finite base field k . This condition was mainly studied by Ramakrishna in [13] and then generalised by Conrad in [4],[5] and Kisin in [9]. From now on, we will only deal with representations of degree 2.

Lemma 3.1. *(Ramakrishna) Let \underline{C} be a full subcategory of $\underline{Rep}_k(G)$ closed under passage to subobjects, direct products and quotients and let \underline{D} be the full subcategory of \underline{P} of pairs (A, V_A) such that $V_A \in \underline{C}$. Then \underline{D} is a deformation condition.*

Proof. We need to prove that \underline{D} satisfies the three properties of deformation conditions. Property 1 comes from the fact that, if $(A, V_A) \rightarrow (B, V_B)$ is a morphism in \underline{D} , then $V_A \otimes_A B \simeq V_B$ and the tensor product can be recovered via direct sums and quotients. Property 2 comes from the closure under subobjects. Property 3 comes from the fact that the fiber product and its projections can be constructed via direct sums and quotients, too. \square

Let F be a finite extension of \mathbb{Q}_p and $\bar{\rho} : G_F \rightarrow GL_2(k)$ be a Galois representation. If ρ is a deformation of $\bar{\rho}$ to a coefficient ring A , we say that ρ is *flat* if there exists a finite flat group scheme X over the ring of integers O_F such that $V_\rho \simeq X(F)$, that is, V_ρ is the generic fiber of X .

Proposition 3.2. *The subcategory \underline{D} of flat deformations is a deformation condition.*

Proof. it suffices to show that \underline{D} satisfies lemma 3.1.

Let $0 \rightarrow T \rightarrow U \rightarrow V \rightarrow 0$ be a sequence of G -modules such that U is the generic fiber of a finite flat group scheme X over O_F . Then we can take the schematic closure X_1 of T in X (see [13, Lemma 2.1] for details) and $X_2 = X/X_1$ to see that also T and V are generic fibers of finite flat group schemes. This argument and the fact that a direct sum of finite flat group schemes is still a finite flat group scheme show that the subcategory of flat deformations is a deformation condition. \square

If $\bar{\rho}$ satisfies the trivial centralizer condition $End_{k[G_F]}(V_{\bar{\rho}}) = k$, then the deformation functor which assigns to a coefficient ring the set of deformations of $\bar{\rho}$ which are flat, called the *flat deformation functor* and denoted as F^{fl} , is representable by a noetherian ring R_p^{fl} , which is called the *local flat universal deformation ring*. We want to give a proof of the main result of representability for this condition, which was proven by Ramakrishna for $p \neq 2$ and by Conrad for all cases. First we need some technical data

Definition 3.3. *Let ϕ denote the absolute Frobenius morphism. A Fontaine-Lafaille module is a $W(k)$ -module M provided with a decreasing, exhaustive, separated filtration of $W(k)$ -submodules $\{M_i\}$ such that, for every index i , there exists a ϕ -semilinear map $\phi_i : M_i \rightarrow M$ with the property that $\phi_i(x) = p\phi_{i+1}(x)$ for every $x \in M$.*

We denote by MF the category of Fontaine-Lafaille modules over $W(k)$. Moreover we denote by MF_{tor}^f the full subcategory of objects such that M has finite length and $\sum \text{Im}(\phi_i) = M$ and by $MF_{tor}^{f,j}$ the subcategory of objects such that $M_0 = M$ and $M_j = 0$. Finally we say that a Fontaine-Lafaille module is *connected* if the morphism ϕ_0 is nilpotent. The main result about Fontaine-Lafaille modules is the following

Theorem 3.4 (Fontaine-Lafaille). *For every $j \leq p$ there exists a faithful exact contravariant functor*

$$MF_{tor}^{f,j} \rightarrow \text{Rep}_{\mathbb{Z}_p}^f(G), \quad (5)$$

which is fully faithful if $j < p$ and becomes fully faithful when restricted to the subcategory of connected Fontaine-Lafaille modules if $j = p$. Moreover $MF_{tor}^{f,2}$ is antiequivalent to the category of finite flat group schemes over $W(k)$

Proof. See [8, Ch.8-9] for a proof and description of the functor. \square

We say that a representation ρ has weight j if it comes from a Fontaine-Lafaille module lying in $MF_{tor}^{f,j}$ and we denote by F_j the subfunctor of $F_{\bar{\rho}}$ given by deformations of $\bar{\rho}$ which are of weight j . It follows that if $\bar{\rho}$ is flat, then the functors F_2 and F_{fl} are the same, therefore we will identify them in the rest of the chapter.

We can now prove the main result for flat deformation functor. The proof is due to Ramakrishna for the case $p > 2$ (see [13, section 3]); then Conrad has shown (see [3]) that the proof works also in the case $p = 2$, since the Fontaine-Lafaille module used is connected.

Theorem 3.5. (Ramakrishna) *Let $\bar{\rho} : G_{\mathbb{Q}_p} \rightarrow GL_2(k)$ be a flat residual Galois representation with trivial centralizer and such that $\det(\bar{\rho}) = \chi$, where χ is the cyclotomic character. Then*

$$R_p^{fl}(\bar{\rho}) \simeq W(k)[[T_1, T_2]]. \quad (6)$$

Proof. We split the proof in two parts. Suppose first that $k = \mathbb{F}_p$ and $\bar{\rho}$ is the representation attached to the p -torsion points of an elliptic curve E over \mathbb{Q}_p with good supersingular reduction. We prove the theorem in this particular case, where computations are relatively easy, and then pass to the general case.

In the particular case we have chosen, we know that $\bar{\rho}$ satisfies the trivial centralizer hypothesis and is of weight 2. We calculate the tangent space $F_2(\mathbb{F}_p[\epsilon])$. Viewing $\mathbb{F}_p[\epsilon]^2$ as a 4-dimensional \mathbb{F}_p -vector space, we can see an element $\rho \in F_2(\mathbb{F}_p[\epsilon])$ as a matrix

$$\rho(g) = \begin{pmatrix} \bar{\rho}(g) & 0 \\ R_g & \bar{\rho}(g) \end{pmatrix} \quad (7)$$

and such a representation gives clearly an element of $\text{Ext}_{2,p}^1(V_{\bar{\rho}}, V_{\bar{\rho}})$, the extensions in the category of weight 2 representations which are killed by p . It is immediate to check that equivalence of liftings correspond to equivalent extensions.

Let M be the Fontaine-Lafaille module associated to $V_{\bar{\rho}}$ via Theorem 3.4. By full faithfulness of the functor, we have that $Ext_{2,p}^1(V_{\bar{\rho}}, V_{\bar{\rho}}) = Ext_{2,p}^1(M, M)$ and that $End_{MF}(M) = \mathbb{F}_p$.

We want to write the module in a compactified manner in terms of a 2×2 -matrix X_M . For that we use the fact that M_1 is 1-dimensional (it will be proved shortly) and that $\phi_0(M_1) = 0$. Then we write

$$\phi_0 = \begin{pmatrix} \alpha & 0 \\ \beta & 0 \end{pmatrix}, \quad \phi_1 = \begin{pmatrix} * & \gamma \\ * & \delta \end{pmatrix}, \quad X_M = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}. \quad (8)$$

The matrix X_M encodes all the informations of the structure of M . We also want to write the elements of $Ext_{2,p}^1(M, M)$ via these matrices. If N is such an element, we have

$$X_N = \begin{pmatrix} X_M & C \\ 0 & X_M \end{pmatrix}, \quad C \in M_2(\mathbb{F}_p). \quad (9)$$

The matrix C corresponds to an element of $Hom(M, M)$. If N' is another element of $Ext_{2,p}^1(M, M)$ and D is the 2×2 matrix in its upper triangular part, then it represents the same extension of N if and only if there exist a matrix $\begin{pmatrix} Id & R \\ 0 & Id \end{pmatrix} \in M_4(\mathbb{F}_p)$ such that

$$\begin{pmatrix} Id & R \\ 0 & Id \end{pmatrix} \begin{pmatrix} X_M & C \\ 0 & X_M \end{pmatrix} = \begin{pmatrix} X_M & D \\ 0 & X_M \end{pmatrix} \begin{pmatrix} Id & R \\ 0 & Id \end{pmatrix} \quad (10)$$

and this happens if and only if $C - D = [R, X_M]$. Moreover R must preserve the filtration of M , because the isomorphism between N and N' does so. Let \mathfrak{H} be the set of such matrices R . It follows that

$$Ext_{2,p}^1(M, M) \simeq Hom(M, M) / \{[R, X_M] : R \in \mathfrak{H}\} \quad (11)$$

Now we know that $dim_{\mathbb{F}_p} M_0 = 2$ and $dim_{\mathbb{F}_p} M_2 = 0$. If $dim_{\mathbb{F}_p} M_1 \neq 1$ then any endomorphism of M does not need to respect any filtration structure and therefore the centralizer of X_M in $M_2(\mathbb{F}_p)$, which has at least dimension 2, would belong to $End_{MF}(M)$; this is impossible because the endomorphism ring is 1-dimensional. Therefore $dim_{\mathbb{F}_p} M_1 = 1$.

Now we can compute the dimension of the tangent space: observe that $Hom(M, M)$ has dimension 4, the set of matrices R which preserves the filtration of M has dimension 3 and the kernel of the map $R \rightarrow [R, X_M]$ has dimension 1 (it is isomorphic to $End_{MF}(M)$). Therefore the tangent space has dimension $4 - (3 - 1) = 2$.

Now we have that $R_2(\bar{\rho}) = \mathbb{Z}_p[[T_1, T_2]]/I$. We count the number of \mathbb{Z}_p/p^l -valued points of the universal ring, which is the number of objects $N \in MF_{tor}^{f,2}$ which are free \mathbb{Z}_p/p^l -modules of rank 2. If N_p denotes the kernel of multiplication by p in N , then we need $N_p \simeq M$, in terms of matrices, since $X_N \equiv X_M \pmod{p}$. Since $X_N \in M_2(\mathbb{Z}_p/p^l)$ and we have to consider modulo p , there are $p^{4(l-1)}$ such matrices. We have to consider them modulo isomorphism. Now if $X_{N_1} \simeq X_{N_2}$, then there exists a matrix $R \in M_2(\mathbb{Z}_p/p^l)$ which respects the

filtration of M such that $RX_{N_1} = X_{N_2}R$; there are $p^{3(l-1)}$ such matrices and p^{l-1} lie in the center of $M_2(\mathbb{Z}_p/p^l)$, therefore commute with all the X_N . So the number of \mathbb{Z}_p/p^l -valued points is $p^{4(l-1)}/(p^{3(l-1)}/p^{l-1}) = p^{2(l-1)}$. Observe that this is the same number of \mathbb{Z}_p/p^l -valued points of $\mathbb{Z}_p[[T_1, T_2]]$.

Let now $f \in I$ and $(x, y) \in (\mathbb{Z}_p/p^l)^2$, then $f(x, y) \equiv 0 \pmod{p^l}$ for every positive integer l . It follows that, taking liftings to characteristic zero, $f(x, y) = 0$ for all $(x, y) \in (p\mathbb{Z}_p)^2$ and therefore $f = 0$. So $I = 0$ and $R_2(\bar{\rho}) = \mathbb{Z}_p[[T_1, T_2]]$.

Now we can pass to the proof of the theorem in the general case and remove the hypothesis that $k = \mathbb{F}_p$ and that $\bar{\rho}$ is the representation coming from an elliptic curve. A lemma of Serre (whose proof can be found in [13]) tells us that $\bar{\rho}$ has restriction to inertia given by

$$\bar{\rho}|_I = \begin{pmatrix} \psi & 0 \\ 0 & \psi^p \end{pmatrix} \quad (12)$$

where ψ is a fundamental character of level 2. For such a representation we will compute both the “unrestricted” universal ring and the flat one. First of all we want to show that $H^2(G, \text{Ad}(\bar{\rho})) = 0$. By Tate local duality we have that $H^2(G, \text{Ad}(\bar{\rho})) = H^0(G, \text{Ad}(\bar{\rho})^*) = (\text{Ad}(\bar{\rho})^*)^G$. Let $\phi \in (\text{Ad}(\bar{\rho})^*)^G$, we want to show that its kernel is 4-dimensional and therefore $\phi = 0$. Let $R \in \text{Ad}(\bar{\rho})$, we have $\phi(gR) = g\phi(R)$, where the G -action is given by conjugacy composed with $\bar{\rho}$ on the left and by determinant on the right. It follows that, if $g \in I$, then $\bar{\rho}(g)R\bar{\rho}(g)^{-1} - \det(\bar{\rho}(g))R \in \text{Ker}(\phi)$. Then, if we define the map

$$T_g : R \rightarrow \bar{\rho}(g)R\bar{\rho}(g)^{-1} - \det(\bar{\rho}(g))R, \quad (13)$$

it suffices to show that there exists $g \in I$ such that $\text{Ker}(T_g) = 0$. We choose a g such that $\psi(g) = \alpha$ where α is an element of order $p^2 - 1$ in k^* . Then, taking explicit formulas

$$R = \begin{pmatrix} x & y \\ z & w \end{pmatrix}, \quad T_g(R) = \begin{pmatrix} x(1 - \alpha^{p+1}) & y(\alpha^{1-p} - \alpha^{p+1}) \\ z(\alpha^{p-1} - \alpha^{p+1}) & w(1 - \alpha^{p+1}) \end{pmatrix} \quad (14)$$

and the last matrix is zero if and only if $R = 0$. Then our claim is proved.

Now we use the formula for the Euler-Poincaré characteristic for $\text{Ad}(\bar{\rho})$. Let $h^i = \dim(H^i(G, \text{Ad}(\bar{\rho})))$. We have

$$c_{EP}(\text{Ad}(\bar{\rho})) = h^0 - h^1 + h^2 = -\dim_k \text{Ad}(\bar{\rho}). \quad (15)$$

We have that $h^2 = 0$, $h^0 = 1$ (it is the trivial centralizer condition) and $\dim_k \text{Ad}(\bar{\rho}) = 4$, therefore $h^1 = 5$. It follows that the unrestricted universal ring for such a representation is isomorphic to $W(k)[[T_1, T_2, T_3, T_4, T_5]]$.

The flat deformation ring can be computed by means of calculations similar to the ones performed in the case $k = \mathbb{F}_p$, except that we have to consider Fontaine-Lafaille modules over $W(k)$ instead of \mathbb{Z}_p and all the dimensions have to be computed over k . We obtain again that $R_2(\bar{\rho}) = W(k)[[T_1, T_2]]$. In particular $R_2(\bar{\rho})$ is a quotient of $R(\bar{\rho})$ and the surjective map between them has a 3-dimensional kernel. The theorem is therefore proved. \square

Now we give a refinement of this result, which is due to Conrad [4].

Theorem 3.6. *Let $\bar{\rho}$ be as in the previous theorem and let $F^{fl,\chi}$ be the subfunctor of flat deformations of $\bar{\rho}$ which have fixed determinant χ . Then this functor is representable by the ring*

$$R_p^{fl,\chi}(\bar{\rho}) \simeq \mathbb{Z}_p[[T]]. \quad (16)$$

Proof. For the proof see [4, Ch. 4, Th.4.1.2]. \square

EXAMPLE: Let E be an elliptic curve over \mathbb{Q}_p that has supersingular reduction in p . Let $\bar{\rho} : G_{\mathbb{Q}_p} \rightarrow GL_2(\mathbb{F}_p)$ be the representation coming from the Galois action on the p -torsion points of E . Then, by the results of [5], $\bar{\rho}$ is absolutely irreducible and therefore the functor $F_{\bar{\rho}}^{fl}$ is representable. Therefore, applying Ramakrishna's theorem, we have that the flat universal deformation ring is $\mathbb{Z}_p[[T_1, T_2]]$.

4 Steinberg representations at primes $\ell \neq p$

Now we want to analyse local conditions at finite primes which are different from p . We continue to assume that the representation space $V_{\bar{\rho}}$ has dimension 2.

Definition 4.1. : *A 2-dimensional representation $\bar{\rho} : G_{\ell} \rightarrow GL_2(k)$ is called of Steinberg type if it is a non-split extension of a character $\lambda : G_{\ell} \rightarrow k^*$ by the twist $\lambda(1) = \lambda \otimes \chi_p$ of λ by the p -adic cyclotomic character χ_p .*

A representation of Steinberg type has the matricial form

$$\bar{\rho}(g) = \begin{pmatrix} \lambda(1)(g) & * \\ 0 & \lambda(g) \end{pmatrix} \quad \forall g \in I_{\ell} \quad (17)$$

Observe that since $\ell \neq p$ the mod p cyclotomic character is unramified and, if p is not a square mod ℓ , it also happens that χ_p and its twists are trivial. We do not impose any ramification restriction on the character λ . Up to twisting by the inverse character of λ , we may assume that $\det(\bar{\rho}) = \chi_p$ and that $V(\bar{\rho})(-1)^{G_{\ell}} \neq 0$, which means that there is a subrepresentation of dimension 1 on which G_{ℓ} acts via χ_p .

We define a subfunctor

$$L_{\bar{\rho}}^{\chi_p} : \underline{Ar} \rightarrow \underline{Sets} \quad (18)$$

of the deformation functor $F_{\bar{\rho}}^{\chi_p}$ as

$$L_{\bar{\rho}}^{\chi_p}(A) = (V_A, L_A) \quad (19)$$

where

- V_A is a deformation of $\bar{\rho}$ to A .

- L_A is a submodule of rank 1 of V_A on which G_ℓ acts via χ_p .

We define in the same way the framed subfunctor $L_{\bar{\rho}}^{\chi_p, \square} : \underline{\hat{A}r} \rightarrow \underline{Sets}$ as

$$L_{\bar{\rho}}^{\chi_p, \square}(A) = (V_A, \beta_A, L_A) \quad (20)$$

where

- (V_A, β_A) is a framed deformation of $\bar{\rho}$ to A .
- L_A is a submodule of rank 1 of V_A on which G_ℓ acts via χ_p .

This is the subfunctor corresponding to liftings of Steinberg type. In the following we work with the framed setting to avoid representability problems.

In order to deal with representability of deformations functors of Steinberg type, we need to recall the main definitions of formal schemes. Let R be a noetherian ring and I an ideal and assume that R is I -adically complete, so that we have

$$R = \varprojlim R/I^n. \quad (21)$$

We define a topological space $Spf(R)$ in the following way: given an element $f \in R$ and \bar{f} its reduction modulo I , we define $D(\bar{f})$ to be the set of prime ideals of R/I not containing \bar{f} . Then the set $Spec(R/I)$ with the induced topology is called the *formal spectrum* of R , with respect to I , and denoted by $Spf(R)$. The sets $D(\bar{f})$ are a basis for the topology of $Spf(R)$.

For each $f \in R$ we define

$$R\langle f^{-1} \rangle = \varprojlim R[f^{-1}]/I^n \quad (22)$$

Then the assignment $D(\bar{f}) \mapsto R\langle f^{-1} \rangle$ defines a structure sheaf on $Spf(R)$.

Definition 4.2. *The affine formal scheme $Spf(R)$ over R with respect to I is the locally ringed space (X, O_X) , where $X = Spec(R/I)$ and $O_X(D(\bar{f})) = R\langle f^{-1} \rangle$ for each $f \in R$.*

A noetherian formal scheme is a locally ringed space (X, O_X) , where X is a topological space and O_X is a sheaf of rings over X such that each point $x \in X$ admits a neighborhood U such that $(U, O_X|_U)$ is isomorphic to an affine formal scheme $Spf(R)$.

A morphism of formal schemes is a pair $(f, f^*) : (X, O_X) \rightarrow (Y, O_Y)$, where $f : X \rightarrow Y$ is a continuous map of topological spaces and $f^* : O_Y \rightarrow f_* O_X$ is a morphism of sheaves.

If (X, O_X) is a scheme, we can obtain a formal scheme \hat{X} by the following construction: let $I \subseteq O_X$ be an ideal sheaf and consider \hat{X} the completion of X along I . Its underlying topological space is given by the subscheme Z of X defined by I and the structure sheaf is defined as before. A formal scheme obtained in this way is called *algebrizable*.

Finally, given a functor $\underline{\hat{A}r} \rightarrow \underline{Sets}$, we can pass to the opposite categories and obtain a functor $\underline{\hat{A}r}^\circ \rightarrow \underline{Sets}^\circ$; $\underline{\hat{A}r}^\circ$ is exactly the category of formal schemes on one point over $\text{Spec}(W(k))$ with residue field $\text{Spec}(k)$. Schlessinger's theorem then provides criteria for the functor to be representable by an object of $\underline{\hat{A}r}^\circ$.

Let us now go back to deformation functors. We want to give a description of the representing object of $L_{\bar{\rho}}^{\chi_p, \square}$ using formal schemes. Let $R = R_{\bar{\rho}}^{\chi_p, \square}$ and V_R be the 2-dimensional module over R with the action given by the universal framed representation $\rho_{\text{univ}}^\square$. Let $\mathbb{P}(R)$ be the projectivization of V_R and $\hat{\mathbb{P}}(R)$ be its completion along the maximal ideal of R . We consider the closed subspace of $\hat{\mathbb{P}}(R)$ defined by the equations $gv - \chi(g)v = 0$ for every $g \in G_\ell$ and each $v \in \hat{\mathbb{P}}(R)$. By formal GAGA, this subspace comes from a unique projective scheme \mathfrak{L} over $\text{Spf}(R)$.

In the following we want to prove some properties of the scheme \mathfrak{L} . In particular we want to show that it is an affine scheme of the form $\text{Spec}(\tilde{R})$ for an appropriate ring \tilde{R} , which will be automatically the representing object of $L_{\bar{\rho}}^{\chi_p, \square}$, because of the defining property of \mathfrak{L} .

Lemma 4.3. *\mathfrak{L} is formally smooth over $\text{Spec}(W(k))$ and its generic fiber $\mathfrak{L} \otimes_{W(k)} W(k)[1/p]$ is connected.*

Proof. Let $A_1 \rightarrow A_2$ be a surjective map in $\underline{\hat{A}r}$. An element $\eta_2 \in L_{\bar{\rho}}^{\chi_p, \square}(A_2)$ corresponds to an extension $c(\eta_2) \in \text{Ext}_{A_2[G]}^1(A_2, A_2(1))$. If η_1 is a lift of η_2 to $L_{\bar{\rho}}^{\chi_p, \square}(A_1)$, then the lift is uniquely determined by a lift of the class $c(\eta_2)$ to an element of $\text{Ext}_{A_1[G]}^1(A_1, A_1(1))$. Finding such a lift of extensions is equivalent to proving that the natural map

$$H^1(G, \mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} M \rightarrow H^1(G, M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1)) \quad (23)$$

is an isomorphism for any $A \in \underline{\hat{A}r}$ and any A -module M . Since H^1 commutes with direct sums, it is sufficient to prove this result for $M = \mathbb{Z}/p^n\mathbb{Z}$. In this case the map is trivially injective and its cokernel is given by $H^2(G, \mathbb{Z}_p(1))[p^n]$; but, by Tate's local duality, $H^2(G, \mathbb{Z}_p(1))$ is the Pontryagin dual of $\mathbb{Q}_p/\mathbb{Z}_p$, which has no p^n -torsion. Therefore the map is an isomorphism.

Now we need to prove connectedness. We denote by $\mathfrak{L}[1/p]$ the generic fiber. By smoothness, the schemes $\mathfrak{L}[1/p]$, \mathfrak{L} and $\mathfrak{L} \otimes_{W(k)} k$ have the same number of connected components and, as schemes over R , the same is true for \mathfrak{L} and $Z = \mathfrak{L} \otimes_R k$; by [10, Prop.2.5.15] this scheme is either all of $\mathbb{P}(k)$, if the action of G_ℓ is trivial, or a single point. Therefore there is only one connected component. \square

Before going on, we need a further notation. Given V a representation lifting $V_{\bar{\rho}_\ell}$ to some ring A , we denote by F_V^χ the subfunctor of $F_{\bar{\rho}_\ell}^\chi$ given by representations lifting V , too, and by $F_V^{\chi, \square}$ the corresponding framed deformation functor.

Lemma 4.4. *The natural morphism of functors $L_V^{\chi, \square} \rightarrow F_V^{\chi, \square}$ is fully faithful. In particular, if V is indecomposable, the morphism is an equivalence, F_V^χ is representable and its tangent space is 0-dimensional.*

Proof. Let $A \in \underline{Ar}$ and (V_A, β_A) be a framed deformation and L_A be a χ_p -invariant line in V_A . We need to show that L_A is unique. Indeed we have $\text{Hom}_{A[G]}(A(1), V_A/L_A)$ is trivial, since $\det(V_A) = \chi$ and V_A/L_A is free of rank 1. Therefore we have $\text{Hom}_{A[G]}(A(1), V_A) = \text{Hom}_{A[G]}(A(1), L_A)$ and the uniqueness follows.

Suppose now that V is indecomposable, then in particular the unframed deformation functor F_V^χ is representable, too. We need to show that each deformation V_A contains an A -line L_A on which G_ℓ acts via χ . For this, it is enough to show that the tangent space is 0-dimensional, which implies that every deformation V_A is isomorphic to $V \otimes_k A$ and therefore inherits the trivial A -line from $V_{\bar{\rho}}$. By Tate's local duality (see Section 3.1) we have

$$h^1(G_\ell, \text{Ad}^0(V)) = h^0(G_\ell, \text{Ad}^0(V)) + h^2(G_\ell, \text{Ad}^0(V)(1)) \quad (24)$$

and that the two summands equal each other; therefore it is enough to show that $h^0(G_\ell, \text{Ad}^0(V)) = 0$. Since $\ell \neq 2$ we have the exact sequence

$$0 \rightarrow H^0(G_\ell, \text{Ad}^0(V)) \rightarrow H^0(G_\ell, \text{Ad}(V)) \rightarrow H^0(G_\ell, \mathbb{Q}_\ell) \rightarrow 0. \quad (25)$$

Trivially $H^0(G_\ell, \mathbb{Q}_\ell) = \mathbb{Q}_\ell$ and $H^0(G_\ell, \text{Ad}(V)) = \mathbb{Q}_\ell$ because V is indecomposable, therefore $H^0(G_\ell, \text{Ad}^0(V)) = 0$ and the lemma is proved. \square

Theorem 4.5. *Let $\text{Spec}(R_{\bar{\rho}}^{\chi,1,\square})$ be the image of the natural morphism $\mathfrak{L} \rightarrow \text{Spec}(R_{\bar{\rho}}^{\chi,\square})$. Then $R_{\bar{\rho}}^{\chi,1,\square}$ is a domain of dimension 4 and $R_{\bar{\rho}}^{\chi,1,\square}[1/p]$ is formally smooth over $W(k)[1/p]$. Moreover, for every $A \in \underline{Ar}$, a morphism $R_{\bar{\rho}}^{\chi,\square} \rightarrow A$ factors through $R_{\bar{\rho}}^{\chi,1,\square}$ if and only if the corresponding 2-dimensional representation is of Steinberg type.*

Proof. The scheme \mathfrak{L} is smooth over $W(k)$ and connected. The ring $R_{\bar{\rho}}^{\chi,1,\square}$ is the ring of global sections of \mathfrak{L} over $R_{\bar{\rho}}^{\chi,\square}$, hence it must be a domain.

If we invert p , lemma 4 tells us that the generic fiber $\mathfrak{L}[1/p]$ is a closed subscheme of $\text{Spec}(R_{\bar{\rho}}^{\chi,\square}[1/p])$, then it must be isomorphic to $\text{Spec}(R_{\bar{\rho}}^{\chi,1,\square}[1/p])$; this proves that $R_{\bar{\rho}}^{\chi,1,\square}[1/p]$ is formally smooth over $W(k)[1/p]$.

We now calculate the dimension. Since $R_{\bar{\rho}}^{\chi,1,\square}$ has no nontrivial p -torsion, it is sufficient to calculate it on the generic fiber and add 1. Let V be an indecomposable point. By lemma 4.4, we have that F_V^χ is representable with tangent space of dimension 0, therefore the framed functor $F_V^{\chi,\square}$ has a tangent space of dimension 3. This proves the claim.

Finally, to prove the last statement, we use again lemma 4.4. A morphism factors through $R_{\bar{\rho}}^{\chi,1,\square}$ if and only if it lifts to a unique point of \mathfrak{L} , that is, if and only if the corresponding representation space V has a 1-dimensional subrepresentation where G acts through χ . The theorem is therefore proved. \square

5 Computations of odd deformation rings

In this section we will deal with local conditions at the infinite places, computing explicitly the deformation ring. Let

$$\bar{\rho}_\infty : \text{Gal}(\mathbb{C}/\mathbb{R}) \rightarrow \text{GL}_2(k) \quad (26)$$

be a local representation at the infinite place with $\det(\bar{\rho}_\infty(\gamma)) = -1 \in k$, where γ is a complex conjugation. Then, up to conjugation, $\bar{\rho}_\infty$ must be of one of these forms

1. $p > 2$, $\bar{\rho}_\infty(\gamma) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.
2. $p = 2$, $\bar{\rho}_\infty(\gamma) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.
3. $p = 2$, $\bar{\rho}_\infty(\gamma) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

A framed deformation of $\bar{\rho}_\infty$ is determined by the image of γ , which is a matrix whose square is the identity and whose characteristic polynomial is $x^2 - 1$. To compute the universal framed ring explicitly, we consider

$$R = W(k)[a, b, c, d]/I. \quad (27)$$

where I is an ideal encoding the condition on the characteristic polynomial. The completion of R to the kernel of the map

$$R \rightarrow k, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow \bar{\rho}_\infty(\gamma) \quad (28)$$

is the universal framed deformation ring of $\bar{\rho}_\infty$.

We do the computation explicitly for the case 3 above (the other two cases being similar). Let M be a lifting of $\bar{\rho}_\infty(\gamma)$ to the universal ring, then, imposing the conditions $\text{Tr}(M) = 0$ and $\det(M) = 1$, we have

$$\begin{aligned} R &= W(k)[a, b, c, d]/((1+a) + (1+d), (1+a)(1+d) + 1 - bc) = \\ &= W(k)[a, b, c]/(-(1+a)^2 + 1 - bc) = \\ &= W(k)[a, b, c]/(-2a - a^2 - bc), \end{aligned} \quad (29)$$

and so $R^\square \simeq W(k)[[a, b, c]]/(2a + a^2 + bc)$. In particular, if we invert p , it is a regular ring of dimension 2 over $W(k)$. A similar computation gives the same result in the other two cases.

6 Local to global arguments

In this chapter we want to give a presentation of a global deformation ring in terms of local ones. The results we use are due to Kisin [9] and will contemplate both the framed and the unframed setting. In the applications, the framed setting is mostly used, to avoid representability problems in the local rings.

Let S be a finite set of primes including p and the infinite prime and Σ a subset of S containing p and the infinite prime too; in many application we will have $\Sigma = S$ and let $\bar{\rho} : G_S \rightarrow GL_2(k)$ be a residual representation.

For each $v \in \Sigma$ we denote by G_v the absolute Galois group of the field \mathbb{Q}_v of v -adic numbers and by $\bar{\rho}_v = \bar{\rho}|_{G_v}$. Assume that $\bar{\rho}$ as well as all of the $\bar{\rho}_v$ for $v \in \Sigma$ satisfy the trivial centralizer hypothesis. Let $F_{\bar{\rho}}$ and $F_{\bar{\rho}_v}$ be the deformation functors associated to $\bar{\rho}$ and $\bar{\rho}_v$ respectively; since all of the functors are representable, we denote by R_v^χ the local universal deformation ring of $\bar{\rho}_v$ with fixed determinant equal to χ and by R_S^χ the universal deformation ring of $\bar{\rho}$ with fixed determinant equal to χ . Finally we put

$$R_\Sigma^\chi = \hat{\otimes}_{v \in \Sigma} R_v^\chi. \quad (30)$$

Let

$$\theta_i : H^i(G_S, \text{Ad}^0(\bar{\rho})) \rightarrow \prod_{v \in \Sigma} H^i(G_v, \text{Ad}^0(\bar{\rho})) \quad (31)$$

be the restriction map in cohomology. Following [9], we denote by r_i and t_i the dimensions of the kernel and cokernel of θ_i as k -vector spaces.

Let m_Σ^χ and m_S^χ be the maximal ideals of R_Σ^χ and R_S^χ respectively and let

$$\eta : m_\Sigma^\chi / ((m_\Sigma^\chi)^2, p) \rightarrow m_S^\chi / ((m_S^\chi)^2, p) \quad (32)$$

be the map between the dual tangent spaces. Then we have the following result

Theorem 6.1. *If the functors $F_{\bar{\rho}}$ and $F_{\bar{\rho}_v}$ are representable, then there exist elements $f_1, \dots, f_{t_1+r_2}$ lying in the maximal ideal of $R_\Sigma^\chi[[x_1, \dots, x_{r_1}]]$ such that*

$$R_S^\chi = R_\Sigma^\chi[[x_1, \dots, x_{r_1}]] / (f_1, \dots, f_{t_1+r_2}). \quad (33)$$

In particular $\dim_{K^{\text{rull}}} R_S^\chi \geq 1 + r_1 - r_2 - t_1$

Proof. Consider the quotient ring R_S^χ / m_Σ^χ ; the tangent space of this ring is clearly the dual of $\ker(\theta_1)$, therefore these two vector spaces have the same dimension. This proves the claim on the number of variables.

Let now $I = \ker(\eta)$. There exists a surjection $R_{gl} = R_\Sigma^\chi[[x_1, \dots, x_{r_1}]] \rightarrow R_S^\chi$ which induces a surjection on tangent spaces with kernel isomorphic to I . Denote by m_{gl} the maximal ideal of R_{gl} and by J the kernel of the surjection. Let ρ_S^χ be the universal deformation of $\bar{\rho}$ and consider a set theoretic lift ρ_{gl} of ρ_S^χ to the ring R_{gl}/Jm_{gl} with determinant χ . Define now a 2-cocycle

$$c : H^2(G_S, J/m_{gl}J \otimes_k \text{Ad}^0(\bar{\rho})), \quad c(g_1, g_2) = \rho_{gl}(g_1 g_2) \rho_{gl}(g_2)^{-1} \rho_{gl}(g_1)^{-1}, \quad (34)$$

where we identify $J/m_{gl}J \otimes_k \text{Ad}^0(\bar{\rho})$ with the kernel of the natural projection map $GL_2(R_{gl}/m_{gl}J) \rightarrow GL_2(R_{gl}/J)$. It is easy to see that the class of c in $H^2(G_S, \text{Ad}^0(\bar{\rho})) \otimes_k J/m_{gl}J$ does not depend on ρ_{gl} , but only on the universal deformation ρ_S^χ and is trivial if and only if ρ_{gl} is a homomorphism.

Now, if we consider the restriction of c to $H^2(G_p, \text{Ad}^0(\bar{\rho}))$, this is the trivial cocycle, because $\rho_S^\chi|_{G_p}$ has a natural lifting to $GL_2(R_{gl})$. Then $c \in \text{Ker}(\theta_2) \otimes_k J/m_{gl}J$. Let $(J/m_{gl}J)^*$ denote the k -dual, then we obtain a map

$$\gamma : (J/m_{gl}J)^* \rightarrow \text{Ker}(\theta_2), \quad \gamma(u) = \langle c, u \rangle; \quad (35)$$

clearly $I^* \subseteq (J/\tilde{m}J)^*$, we claim that $\text{Ker}(\gamma) \subseteq I^*$.

Let $u \in \text{Ker}(\gamma)$ be a nonzero element; we denote by R_{gl}^u the push-out of $R_{gl}/m_{gl}J$ by u , so that $R_S^\chi \simeq R_{gl}^u/I^u$, with I^u an ideal of square zero and isomorphic to k as an R_{gl}^u -module. Since $u \in \text{Ker}(\gamma)$ we can find a representation $\rho_u : G_{\mathbb{Q},S} \rightarrow GL_2(\tilde{R}_u)$ with determinant χ which lifts ρ_S^χ . Then, by the universal property of R_S^χ the natural map $R_{gl}^u \rightarrow R_S^\chi$ has a section; it follows that $R_{gl}^u \simeq R_S^\chi \oplus I^u$ and $R_{gl}^u/pR_{gl}^u \simeq R_S^\chi/pR_S^\chi \oplus I_u$. Therefore the map $R_{gl}^u \rightarrow R_S^\chi$ does not reduce to an isomorphism on tangent spaces and it follows that the induced map

$$\text{Ker}(J/m_{gl}J \rightarrow I) \rightarrow J/m_{gl}J \rightarrow I^u \quad (36)$$

is not surjective and must be the zero map, that is, u factors through I and we have proved the claim.

Hence we have proved that

$$\dim(J/m_{gl}J) = \dim_k \text{Ker}(\gamma) + \dim_k \text{Im}(\gamma) \leq \dim(I) + r_2 = t_1 + r_2 \quad (37)$$

and we are done. \square

The hypotheses of the theorem are too strong for concrete applications, because they require all the functors to be representable. Therefore we want to establish a similar result in the framed setting. For the rings and ideals we have already defined, we simply add the \square superscript to indicate that we are in the framed case

We need to define an auxiliary functor

$$F_{\Sigma,S}^{\chi,\square} : \hat{Ar} \rightarrow \underline{Sets} \quad (38)$$

which associates to every coefficient ring A a deformation of $\bar{\rho}$ to A and a Σ -tuple of bases of V_A in the following way:

$$F_{\Sigma,S}^{\chi,\square}(A) = \{(V_A, \iota_A, (\beta_v)_{v \in \Sigma}) \mid (V_A, \iota_A) \in F_{\bar{\rho}}^\chi(A), \iota_A(\beta_v) = \beta \ \forall v \in \Sigma\} / \simeq. \quad (39)$$

We have natural morphisms of functors

$$\begin{array}{ccc} F_{\Sigma,S}^{\chi,\square} & \longrightarrow & \prod_{v \in \Sigma} F_{\rho_v} \\ \downarrow & & \\ F_{\bar{\rho}}^\chi & & \end{array}$$

where the horizontal map is the restriction modulo each $v \in \Sigma$ and the vertical map is simply the forgetful functor which ignores bases. The following proposition describes the nature of these morphisms.

Proposition 6.2. *The natural morphism $F_{\Sigma,S}^{\chi,\square} \rightarrow F_{\bar{\rho}}^\chi$ is smooth and, passing to universal ring, we have an isomorphism*

$$R_{\Sigma,S}^{\chi,\square} \simeq R_S^\chi[[x_1, \dots, x_{4|\Sigma|-1}]]. \quad (40)$$

Moreover the morphism $F_{\Sigma, S}^{\chi, \square} \rightarrow \prod_{v \in \Sigma} F_{\rho_v}$ gives a homomorphism of universal rings

$$R_{loc} = \hat{\otimes}_{v \in \Sigma} R_v^{\chi, \square} \rightarrow R_{\Sigma, S}^{\chi, \square}. \quad (41)$$

Proof. The smoothness and the dimension formula come from the smoothness of the framed deformation functor over the unframed one and the morphism of universal rings comes naturally from the morphism of functors. \square

The passage to local rings is the key for computing $R_{\Sigma, S}^{\chi, \square}$. The use of framed deformations avoids the representability issues.

We can now state one of the main results of this approach. We need a generalization of the map θ_1 , defined at the beginning of the chapter

Lemma 6.3 (Key lemma). *Let*

$$\theta_1^{\square} : F_{\Sigma, S}^{\chi, \square}(k[\epsilon]) \rightarrow \bigoplus_{v \in \Sigma} F_{\rho_v}^{\chi, \square}(k[\epsilon]) \quad (42)$$

be the restriction map on tangent spaces and set $r = \dim_k \text{Ker}(\theta_1^{\square})$ and $t = \dim_k \text{Ker}(\theta_2) + \dim_k \text{coKer}(\theta_1^{\square})$. Then we have a presentation

$$R_{\Sigma, S}^{\chi, \square} \simeq R_{loc}[[x_1, \dots, x_r]]/(f_1, \dots, f_t) \quad (43)$$

Proof. The proof is the same of theorem 6.1 in the unframed setting, simply substituting R_S^{χ} with $R_{\Sigma, S}^{\chi, \square}$ and the cohomological groups and the map θ with their framed counterparts. \square

Observe that r is an optimal value, while t is just an upper bound on the number of relations; for example some of the f_i may be trivial.

Now we need also to evaluate $\delta = \dim_k \text{coKer}(\theta_2)$. Note that θ_2 is part of the Poitou-Tate sequence (see [19] for references) and that $H^2(G_v, \text{Ad}^0(\bar{\rho})) \simeq H^0(G_v, \text{Ad}^0(\bar{\rho})^*)^*$ by local Tate duality. Therefore

$$\begin{aligned} \delta &= \dim_k \text{coKer}(\theta_2) = \\ &= \dim_k \text{Ker}(H^0(G_S, \text{Ad}^0(\bar{\rho})^*) \rightarrow \bigoplus_{v \in S \setminus \Sigma} H^0(G_v, \text{Ad}^0(\bar{\rho})^*)). \end{aligned} \quad (44)$$

Note that $\delta = 0$ if $S \setminus \Sigma$ is non-empty, and therefore contains a finite prime, or if the image of $\bar{\rho}$ is non-solvable, and therefore $H^0(G_S, \text{Ad}^0(\bar{\rho})^*)$ is trivial.

The following result gives us a link between all the quantities we have defined

Theorem 6.4. *If Σ contains all the places above p and ∞ , then $r - t + \delta = |\Sigma| - 1$.*

Proof. We will make use of the Tate's computation of the Euler-Poincaré characteristic (a proof of which can be found in [19])

$$c_{EP}(G_S, \text{Ad}^0(\bar{\rho})) = -\dim_k \text{Ad}^0(\bar{\rho}) + h^0(G_{\infty}, \text{Ad}^0(\bar{\rho}_{\infty})) \quad (45)$$

and the local version

$$c_{EP}(G_v, Ad^0(\bar{\rho}_v)) = \begin{cases} -\dim_k Ad^0(\bar{\rho}_v) & \text{if } v = p \\ h^0(G_v, Ad^0(\bar{\rho}_v)) & \text{if } v = \infty \\ 0 & \text{otherwise} \end{cases} \quad (46)$$

Then we have

$$\begin{aligned} r - t + \delta &= \dim_k Ker(\theta_1^\square) - \dim_k coKer(\theta_1^\square) - \dim_k Ker(\theta_2) + \\ &+ \dim_k coKer(\theta_2) = \dim_k F_{\Sigma, S}^{\chi, \square}(k[\epsilon]) - \sum_{v \in \Sigma} \dim_k F_v(k[\epsilon]) - h^2(G_S, Ad^0(\bar{\rho})) + \\ &\quad + \sum_{v \in \Sigma} h^2(G_v, Ad^0(\bar{\rho}_v)). \end{aligned} \quad (47)$$

Now we evaluate the dimensions of tangent spaces and we have

$$\begin{aligned} h^1(G_S, Ad^0(\bar{\rho})) - h^0(G_S, Ad^0(\bar{\rho})) - 1 + |\Sigma|n^2 - h^2(G_S, Ad^0(\bar{\rho})) + \\ - \sum_{v \in \Sigma} (h^1(G_v, Ad^0(\bar{\rho}_v)) - h^0(G_v, Ad^0(\bar{\rho}_v)) - 1 + n^2 - h^2(G_v, Ad^0(\bar{\rho}_v))) = \\ = -c_{EP}(G_S, Ad^0(\bar{\rho})) + \sum_{v \in \Sigma} c_{EP}(G_v, Ad^0(\bar{\rho}_v)) + |\Sigma| - 1. \end{aligned} \quad (48)$$

Finally we use Tate's formulas for c_{EP} and we have

$$\begin{aligned} \dim_k Ad^0(\bar{\rho}) - h^0(G_\infty, Ad^0(\bar{\rho}_\infty)) - \dim_k Ad^0(\bar{\rho}_p) + \\ + h^0(G_\infty, Ad^0(\bar{\rho}_\infty)) + |\Sigma| - 1 = |\Sigma| - 1. \end{aligned} \quad (49)$$

□

7 Geometric deformation rings

In this chapter we want to give some results about a particular class of deformation problems. We suppose that our $\bar{\rho}$ is odd of dimension 2 and absolutely irreducible (so that the deformation functor is representable). The rest of the notation matches the one of the previous chapter.

For each $v \in \Sigma$ let $\tilde{F}_v^{\chi, \square}$ be a representable subfunctor of $F_v^{\chi, \square}$ such that the corresponding representing ring $\tilde{R}_v^{\chi, \square}$ (which is a quotient of $R_v^{\chi, \square}$) is flat over \mathbb{Z}_p and satisfies:

$$\dim_{K_{rull}} \tilde{R}_v^{\chi, \square}[1/p] = \begin{cases} 3 & \text{if } v \neq p, \infty \\ 4 & \text{if } v = p \\ 2 & \text{if } v = \infty \end{cases} \quad (50)$$

A deformation functor satisfying these properties will be called a *geometric deformation functor*. The name was given by Kisin to match the results on *geometric* representations defined in Fontaine-Mazur's Conjecture (See [10] for details) which all share this property. Such a functor satisfies the following properties:

- $\tilde{R}_{loc} = \hat{\otimes}_{v \in \Sigma} \tilde{R}_v^{\chi, \square}$ is flat over \mathbb{Z}_p and its Krull dimension is $\geq 3|\Sigma| + 1$.
- The functors \tilde{F}_S^χ and $\tilde{F}_{\Sigma, S}^{\chi, \square}$ are representable.
- The ring $\tilde{R}_{\Sigma, S}^{\chi, \square}$ is isomorphic to $R_{\Sigma, S}^{\chi, \square} \hat{\otimes}_{v \in \Sigma} \tilde{R}_{loc}$ and therefore

$$\tilde{R}_{\Sigma, S}^{\chi, \square} \simeq \tilde{R}_{loc}[[x_1, \dots, x_r]]/(f_1, \dots, f_t) \quad (51)$$

with r, t defined as before. In particular the Krull dimension of $\tilde{R}_{\Sigma, S}^{\chi, \square} \geq 4|\Sigma| - \delta$.

Since the map $\tilde{F}_{\Sigma, S}^{\chi, \square} \rightarrow \tilde{F}_S^\chi$ is smooth, we can obtain the following result

Theorem 7.1. *If $\delta = 0$, then $\dim_{Krull} \tilde{R}_S^\chi \geq 1$.*

8 The main result

Now we have all the necessary instruments to generalize the results of the previous chapter. Let $\bar{\rho}_1, \dots, \bar{\rho}_n$ be representations of $G_{\mathbb{Q}}$ each with values in $GL_2(k)$, where k is a finite field of characteristic p . Let \underline{Gr} be the category of finite flat group scheme over \mathbb{Z}_p of order a power of p and let \underline{D} be a subcategory of \underline{Gr} closed by products, subobjects and quotients. We assume that each $V_{\bar{\rho}_i}$ is the generic fiber of an element of \underline{D} . We write

$$\bar{\rho} = \bar{\rho}_1 \oplus \dots \oplus \bar{\rho}_n : G_{\mathbb{Q}} \rightarrow GL_{2n}(k). \quad (52)$$

It may happen that some of the $\bar{\rho}_i$ are isomorphic. Therefore we suppose that there are exactly r different representations among the $\bar{\rho}_i$ which are non-isomorphic and we assume them to be $\bar{\rho}_1, \dots, \bar{\rho}_r$. Then we rewrite $\bar{\rho}$ as

$$\bar{\rho} = \bigoplus_{i=1}^r \bar{\rho}_i^{e_i}. \quad (53)$$

We want to define a deformation functor in this case. We start considering the single representation $\bar{\rho}_i$. We define the deformation functor $F_{\bar{\rho}_i, \underline{D}} : \underline{Ar} \rightarrow \underline{Sets}$ which sends an artinian ring A to the set of deformation classes ρ_i of $\bar{\rho}_i$ to A such that

- ρ_i is p -flat over $\mathbb{Z}[1/\ell]$;
- ρ_i satisfies $(\rho_i(g) - Id)^2 = 0$ for every $g \in I_{\ell}$;
- ρ_i is odd;

and let $F_{\bar{\rho}, \underline{D}} : \underline{Ar} \rightarrow \underline{Sets}$ be the deformation functor associated to $\bar{\rho}$ with the same local conditions.

Lemma 8.1. $F_{\bar{\rho}_i, \underline{D}}$ is a geometric deformation functor.

Proof. We need to show that our local conditions satisfy the definition of geometric functor defined in section 7. At the prime p we apply theorem 3.6 which tells us that the local ring is isomorphic to $\mathbb{Z}_p[[X]]$; in particular, after inverting p its framed counterpart has Krull dimension 4, as in the geometric conditions. At the infinite prime, the computations of section 5 tells us that the dimension over \mathbb{Z}_p of the framed deformation ring is 2. Finally at the prime ℓ the condition that $(\rho_i(\sigma) - id)^2 = 0$ is equivalent to a Steinberg type condition with λ equal to the trivial character. Therefore theorem 4.5 gives us that the framed deformation ring has Krull dimension 4; in particular, inverting p it is regular of dimension 3. It follows that all the conditions of being a geometric deformation functor are satisfied. Then we can apply theorem 7.1 and obtain that each $F_{\bar{\rho}_i}$ has a representing ring of Krull dimension at least 1. \square

Theorem 8.2 (Main theorem: dimension 2 case). *Suppose that:*

1. $Ext_{\underline{D}, p}^1(V_{\bar{\rho}_i}, V_{\bar{\rho}_j})$ of killed-by- p extensions is trivial for every $i, j = 1, \dots, r$;
2. $Hom_G(V_{\bar{\rho}_i}, V_{\bar{\rho}_j}) = \begin{cases} k & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$.

Then the functor $F_{\bar{\rho}, \underline{D}}^\square$ is represented by a power series ring over $W(k)$ in N variables, where

$$N = 4n^2 - \sum_{i=1}^r e_i^2. \quad (54)$$

Proof. The representation $\bar{\rho}$ has the following matrix form

$$\begin{pmatrix} \begin{pmatrix} \bar{\rho}_1 & & \\ & \ddots & \\ & & \bar{\rho}_1 \end{pmatrix} & & \\ & \ddots & \\ & & \begin{pmatrix} \bar{\rho}_r & & \\ & \ddots & \\ & & \bar{\rho}_r \end{pmatrix} \end{pmatrix}. \quad (55)$$

We call \bar{T} this matrix and $\bar{\beta}$ a k -basis of $V_{\bar{\rho}}$ in which $\bar{\rho}$ has this matrix form; \bar{T} belongs to $M_h(k)$ where we denote by $h = 2n$. We also denote by $h_j = \sum_{i=1}^{j-1} 2e_i$.

By the lemma, we know that each $\bar{\rho}_i$ is geometric, therefore the functor $F_{\bar{\rho}_i}$ is represented by a ring of Krull dimension ≥ 1 ; on the other hand the hypothesis of triviality of extension set tells us that the tangent space of $F_{\bar{\rho}_i}$ is trivial, therefore the universal deformation ring is a quotient of $W(k)$. Therefore the universal ring must be isomorphic to $W(k)$ and there exist a p -adic lift of $\bar{\rho}_i$, given by the universal representation, that we call ρ_i .

Let then T be the matrix obtained by \bar{T} replacing all the $\bar{\rho}_i$ with the respective ρ_i , V_ρ the associated representation module over $W(k)$ and β a basis of V_ρ lifting $\bar{\beta}$ in which T has the block-diagonal shape. We look for a framed deformation of \bar{T} of the form

$$\tilde{T} = (1 + M(\underline{x}))T(1 + M(\underline{x}))^{-1} \quad (56)$$

where $M = M(\underline{x})$ is the matrix having a variable $x_{i,j}$ as (i, j) -th entry and \underline{x} is the array of all such x . We write M as

$$\begin{pmatrix} M_{1,1} & M_{1,2} & \dots & M_{1,r} \\ M_{2,1} & M_{2,2} & & \vdots \\ \vdots & & \ddots & \\ M_{r,1} & M_{r,2} & \dots & M_{r,r} \end{pmatrix}. \quad (57)$$

where $M_{i,j}$ is the $2e_i \times 2e_j$ submatrix given by

$$M_{i,j} = \begin{pmatrix} x_{h_i+1, h_j+1} & \dots & x_{h_i+1, h_j+1} \\ \vdots & \ddots & \vdots \\ x_{h_i+1, h_j+1} & \dots & x_{h_i+1, h_j+1} \end{pmatrix}. \quad (58)$$

Then we have that $(\tilde{T}, \beta(1 + M))$ gives a framed deformation of $\bar{\rho}$ to the ring $R = W(k)[[x_{1,1}, \dots, x_{2n,2n}]]$.

We want to modify our deformation by a linear transformations lying in the centralizer of ρ to kill some of the variables. Consider the diagonal submatrices $M_{i,i}$; we can eventually subdivide it in 2×2 submatrices

$$\begin{pmatrix} M_{i,i}^{(1,1)} & \dots & M_{i,i}^{(1,e_i)} \\ \vdots & \ddots & \vdots \\ M_{i,i}^{(e_i,1)} & \dots & M_{i,i}^{(e_i,e_i)} \end{pmatrix}, \quad (59)$$

where

$$M_{i,i}^{(s,t)} = \begin{pmatrix} x_{h_i+2s-1, h_i+2t-1} & x_{h_i+2s-1, h_i+2t} \\ x_{h_i+2s, h_i+2t-1} & x_{h_i+2s, h_i+2t} \end{pmatrix}. \quad (60)$$

We look for a matrix $Y \in M_h(R)$ such that $1 + Y$ commutes with T and the conjugation by $1 + Y$ does not modify the framed deformation class. The hypothesis 2 on the mutual endomorphisms of the ρ_i implies that Y must be a block diagonal matrix of the form

$$Y = \text{diag}[Y_1, \dots, Y_r] \quad (61)$$

where $Y_i = A_i \otimes id_2 \in M_{2e_i}(R)$ and $A_i \in M_{e_i}(R)$.

Now we need to choose properly the entries $\{a_{ist}\}_{s,t=1,\dots,e_i}$ of the matrices Y_i . Let

$$(1 + M)(1 + Y)T(1 + Y)^{-1}(1 + M)^{-1} = (1 + \tilde{M})T(1 + \tilde{M})^{-1}. \quad (62)$$

We set

$$a_{ist} = \frac{-x_{h_i+2s, h_i+2t}}{1 + x_{h_i+2s, h_i+2t}}. \quad (63)$$

The resulting matrix \tilde{M} has entries $\tilde{x}_{u,v}$ given by

$$\tilde{x}_{u,v} = \begin{cases} 0 & \text{if } u = h_i + 2s, v = h_i + 2t \\ \frac{x_{u,v}}{1+x_{h_i+2s, h_i+2t}} & \text{otherwise} \end{cases} \quad (64)$$

To make the notation easier, we rename $\tilde{M} = M$ and $\tilde{x}_{i,j} = x_{i,j}$. We call $(\tilde{\rho}, \tilde{\beta}(1+Y))$ the resulting framed deformation obtained at the end of this process.

The framed deformation $\tilde{\rho}$ has values in the ring

$$\tilde{R} = W(k)[[x_{1,1}, \dots, x_{2n,2n}]] / (x_{h_i+2s, h_i+2t} : s, t = 1, \dots, e_i, i = 1, \dots, r) \quad (65)$$

We need to show that this is effectively the universal framed deformation. Observe that \tilde{R} is a power series ring over $W(k)$ in exactly N variables. First we need to compute the dimension of the framed tangent space. We use the fact that the tangent space $F_{\tilde{\rho}, S}^\square(k[\epsilon])$ fits the exact sequence

$$0 \rightarrow F_{\tilde{\rho}, S}(k[\epsilon]) \rightarrow F_{\tilde{\rho}, S}^\square(k[\epsilon]) \rightarrow \text{Ad}(\tilde{\rho}) / \text{Ad}(\tilde{\rho})^G \rightarrow 0 \quad (66)$$

and that the unframed tangent space is trivial, because of the triviality of the extension set. Note that

$$\text{Ad}(\tilde{\rho})^G = \text{End}_G(\tilde{\rho}_1^{e_1} \oplus \dots \oplus \tilde{\rho}_r^{e_r}) = \bigoplus_{i=1}^r \text{End}_G(\tilde{\rho}_i^{e_i}) = \bigoplus_{i=1}^r M_{e_i}(k) \quad (67)$$

where we have used the hypothesis on the sets $\text{Hom}_G(V_{\tilde{\rho}_i}, V_{\tilde{\rho}_j})$.

Therefore we have

$$\dim(F_{\tilde{\rho}, S}^\square(k[\epsilon])) = \dim(\text{Ad}(\tilde{\rho})) - \dim(\text{Ad}(\tilde{\rho})^G) = 4n^2 - \sum_{i=1}^r e_i^2 = N, \quad (68)$$

then the universal framed deformation ring $R_{\tilde{\rho}, S}^\square$ and \tilde{R} have the same relative Krull dimension.

Now we use the universality of $R_{\tilde{\rho}, S}^\square$ that gives us a unique $W(k)$ -algebra morphism $\pi : R_{\tilde{\rho}, S}^\square \rightarrow \tilde{R}$ such that $\hat{\pi} \circ \rho_{\text{univ}} = \tilde{\rho}$ where ρ_{univ} is the universal representation and $\hat{\pi}$ is the extension of π to GL_2 . We have a diagram

$$\begin{array}{ccc} W(k)[[x_1, \dots, x_N]]^1 & \xrightarrow{\pi_1} & R_{\tilde{\rho}, S}^\square \\ & \searrow \pi_2 & \downarrow \pi \\ & & \tilde{R} \end{array} \quad \simeq W(k)[[y_1, \dots, y_N]].$$

If the map π is surjective, since π_1 is surjective, too, it follows that π_2 is surjective, too. But π_2 is $W(k)$ -algebra map between algebras of the same dimension and therefore it must be an isomorphism. But then π must be an isomorphism, too. The theorem is therefore proved, provided that π is surjective.

To prove that the map π is surjective, it is enough to show that the induced map on mod p tangent space

$$\tilde{\pi} : \text{Hom}(\tilde{R}/p, k[\epsilon]) \rightarrow \text{Hom}(R_{\tilde{\rho}, S}^\square/p, k[\epsilon]) \quad (69)$$

is injective (because the functor $\text{Hom}(\cdot, k[\epsilon])$ is contravariant). Since \tilde{R} is a power series ring over $W(k)$ in N variables, an element of $\text{Hom}(\tilde{R}/p, k[\epsilon])$ is given by a map which sends the variables x_1, \dots, x_N to elements $\epsilon\alpha_1, \dots, \epsilon\alpha_N$ with $\alpha_1, \dots, \alpha_N$ giving a basis for the $k[\epsilon]$ -module V given by a representation lifting $V_{\bar{\rho}}$; different elements are given by different choices of the basis. Suppose then that two elements $(V, \{\alpha_i\}), (V, \{\alpha_j\})$ have the same image with respect to $\tilde{\pi}$; it means that there exists a matrix $A \in GL_n(k[\epsilon])$ whose conjugation maps the basis $\{\alpha_i\}$ into $\{\alpha_j\}$ and A commutes with the representation, that is, lies in the centralizer of the image of $\bar{\rho}$. But, because of the construction of the representation $\bar{\rho}$ such matrix must be the identity. Therefore the map is injective and the theorem is proved. \square

REMARK: The existence of a p -adic lift to $W(k)$ for each $\bar{\rho}_i$ can be proved even if do not have the triviality of the extension set, in the case p odd, if we had the condition that

$$\bar{\rho}_i|_{G_{\mathbb{Q}(\sqrt{\mp p})}}. \quad (70)$$

This condition is automatically implied by the absolute irreducibility of $\bar{\rho}_i$ if $p > 3$ (see [9, Ch.4.2] for a proof). The case $p = 3$ is treated in [6, Lemma 3.1 and 3.2].

As an application of the theorem, we recover an example of Schoof in [15]. Consider an abelian variety A over \mathbb{Q} which has good reduction in all but one prime ℓ , where it has semistable reduction. By [15, Th. 1.2], if $\ell = 11$ then A is isogenous to a product of copies of $E = J_0(11)$. Moreover A is supersingular at 2 and $A[2] \simeq E[2]^g$. Therefore we can look at the natural $G_{\mathbb{Q}}$ -representation $\bar{\rho}_{A,2}$ on the 2-torsion points of A as product of g copies of the representation $\bar{\rho}_{E,2}$. In formulas

$$\bar{\rho}_{A,2} = \bigoplus_{i=1}^g \bar{\rho}_{E,2}. \quad (71)$$

Then we can study the deformations of $\bar{\rho}_{A,2}$ from the ones of $\bar{\rho}_{E,2}$. Applying the theorem we have that the functor $F_{\bar{\rho}_{A,2}, \underline{D}}$ is represented by a power series ring over \mathbb{Z}_p in $3g^2$ variables. Moreover, if we go through the same construction as in the theorem, we have that the universal framed deformation is given taking the product of g copies of the \mathbb{Z}_p -representation given by the Tate module $T_p E$ and then applying the transformation with the matrix M .

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